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An algorithm for computing the cumulative
eccentric Fisher distribution

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Abstract

The paper gives an algorithm for computing the cumulative eccentric Fisher distribution. The method is exact for the central Fisher distribution, and when the eccentricity differs from zero, we can get the inaccuracy less than an arbitrarily chosen $\epsilon > 0$. The algorithm is programmed on Hewlett-Packard HP-67 pocket calculator, and as a by-product we have included a routine for the cumulative central Student distribution.

1. Some relations between distribution functions

1A. For $\alpha, \beta > 0$ let $B(\cdot; \alpha, \beta)$ denote the cumulative β -distribution with parameters (α, β) , that is,

$$B(x; \alpha, \beta) = \int_0^x \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1} dt \quad 0 \leq t \leq 1 \quad (1)$$

and let

$$b(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha}(1-x)^{\beta} \quad 0 \leq x \leq 1 \quad (2)$$

It is easily seen that

$$B(x; \alpha, \beta) = 1 - B(1-x; \beta, \alpha) \quad (3)$$

$$b(x; \alpha, \beta) = b(1-x; \beta, \alpha) \quad (4)$$

Furthermore, from (2) we see that

$$b(x; \alpha+1, \beta) = \frac{\alpha+\beta}{\alpha} x b(x; \alpha, \beta) \quad (5)$$

$$b(x; \alpha, \beta+1) = \frac{\alpha+\beta}{\beta} (1-x) b(x; \alpha, \beta) \quad (6)$$

$$b(x; \alpha, 1) = \alpha x^{\alpha}(1-x) \quad (7)$$

$$b(x; 1, \beta) = \beta x(1-x)^{\beta} \quad (8)$$

$$b(x; \frac{1}{2}, \frac{1}{2}) = \frac{\sqrt{x(1-x)}}{\pi}, \quad (9)$$

and these expressions may be used for recursive computation of $b(x; \alpha, \beta)$.

From (1) we get

$$B(x; \alpha, \beta) - B(x; \alpha+1, \beta) =$$

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} \int_0^x [\alpha t^{\alpha-1}(1-t)^{\beta} - \beta t^{\alpha}(1-t)^{\beta-1}] dt = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} \int_0^x \frac{d}{dt} [t^{\alpha}(1-t)^{\beta}] dt =$$

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} x^{\alpha}(1-x)^{\beta} = \frac{b(x; \alpha, \beta)}{\alpha},$$

that is,

$$B(x; \alpha+1, \beta) = B(x; \alpha, \beta) - \frac{b(x; \alpha, \beta)}{\alpha}. \quad (10)$$

By (3), (4), and (10) we obtain

$$B(x; \alpha, \beta+1) = B(x; \alpha, \beta) + \frac{b(x; \alpha, \beta)}{\beta} . \quad (11)$$

By putting $\alpha = \beta = \frac{1}{2}$ in (3) we obtain

$$\begin{aligned} B(x; \frac{1}{2}, \frac{1}{2}) &= \int_0^x \frac{1}{\pi} \frac{1}{\sqrt{t(1-t)}} dt = \frac{1}{\pi} \int_0^x \frac{2}{\sqrt{1-(1-2t)^2}} dt = \\ &= \frac{1}{\pi} \int_0^x -\frac{d}{dt} \arcsin(1-2t) dt = \frac{1}{2} - \frac{\arcsin(1-2x)}{\pi} \end{aligned}$$

that is,

$$B(x; \frac{1}{2}, \frac{1}{2}) = \frac{1}{2} - \frac{\arcsin(1-2x)}{\pi} . \quad (12)$$

From (1) we easily find

$$B(x; \alpha, 1) = x^\alpha \quad (13)$$

$$B(x; 1, \beta) = 1-(1-x)^\beta . \quad (14)$$

1B. For $\theta > 0$ let $p(\cdot; \theta)$ denote the Poisson density with parameter θ , that is,

$$p(x; \theta) = \frac{\theta^x}{x!} e^{-\theta} \quad x = 0, 1, 2, \dots .$$

It is easily seen that $p(x; \theta)$ may be computed recursively by

$$p(x+1; \theta) = \frac{\theta}{x+1} p(x; \theta) \quad (15)$$

$$p(0; \theta) = e^{-\theta} . \quad (16)$$

1C. For $\lambda \geq 0$ and $\mu, v = 1, 2, 3, \dots$ let $G_{\mu, v}(\cdot; \lambda)$ denote the cumulative eccentric Fisher distribution with μ and v degrees of freedom and eccentricity λ . From formula (2.41) on page 29 in Sverdrup (1967) we easily obtain

$$G_{\mu, v}(f; \lambda) = \sum_{j=0}^{\infty} a_j \quad (17)$$

with

$$a_j = B\left(\frac{f\mu}{v+f\mu}; \frac{\mu}{2}+j, \frac{v}{2}\right) P\left(j; \frac{\lambda}{2}\right) . \quad (18)$$

The following theorem (Sverdrup (1976)) gives an upper bound on the inaccuracy we get by adding only a finite number of a_j 's in (17).

Theorem 1. Let $m > \frac{\lambda}{2} - 1$. Then

$$\sum_{j=m+1}^{\infty} a_j < a_m \frac{\frac{\lambda}{2}}{m+1-\frac{\lambda}{2}}$$

Proof. By (10) we see that $B(x; \alpha+1, \beta) < B(x; \alpha, \beta)$.

Hence

$$\begin{aligned} \sum_{j=m+1}^{\infty} a_j &= \sum_{j=m+1}^{\infty} B\left(\frac{f\mu}{v+f\mu}; \frac{\mu}{2}+j, \frac{v}{2}\right) p(j; \frac{\lambda}{2}) < B\left(\frac{f\mu}{v+f\mu}; \frac{\mu}{2}+m, \frac{v}{2}\right) \sum_{j=m+1}^{\infty} p(j; \frac{\lambda}{2}) = \\ a_m \sum_{j=m+1}^{\infty} \frac{(\frac{\lambda}{2})^{j-m}}{j(j-m)} &< a_m \sum_{j=m+1}^{\infty} \left(\frac{\lambda}{2(m+1)}\right)^{j-m} = a_m \frac{\frac{\lambda}{2}}{m+1-\frac{\lambda}{2}}. \end{aligned} \quad \text{Q.E.D.}$$

The following corollary follows trivially from the theorem.

Corollary 1. Let $\epsilon > 0$, and let m be the smallest integer such that

$$a_m < \left(\frac{m+1}{\frac{\lambda}{2}} - 1\right) \epsilon. \quad (19)$$

Then

$$G_{\mu, v}(f; \lambda) - \epsilon < \sum_{j=0}^m a_j < G_{\mu, v}(f; \lambda).$$

1D. Let T be a random variable, centrally Student distributed with v degrees of freedom, and let T_v denote the cumulative distribution of T . As T is symmetricly distributed about zero and T^2 is centrally Fisher distributed with 1 and v degrees of freedom, we have

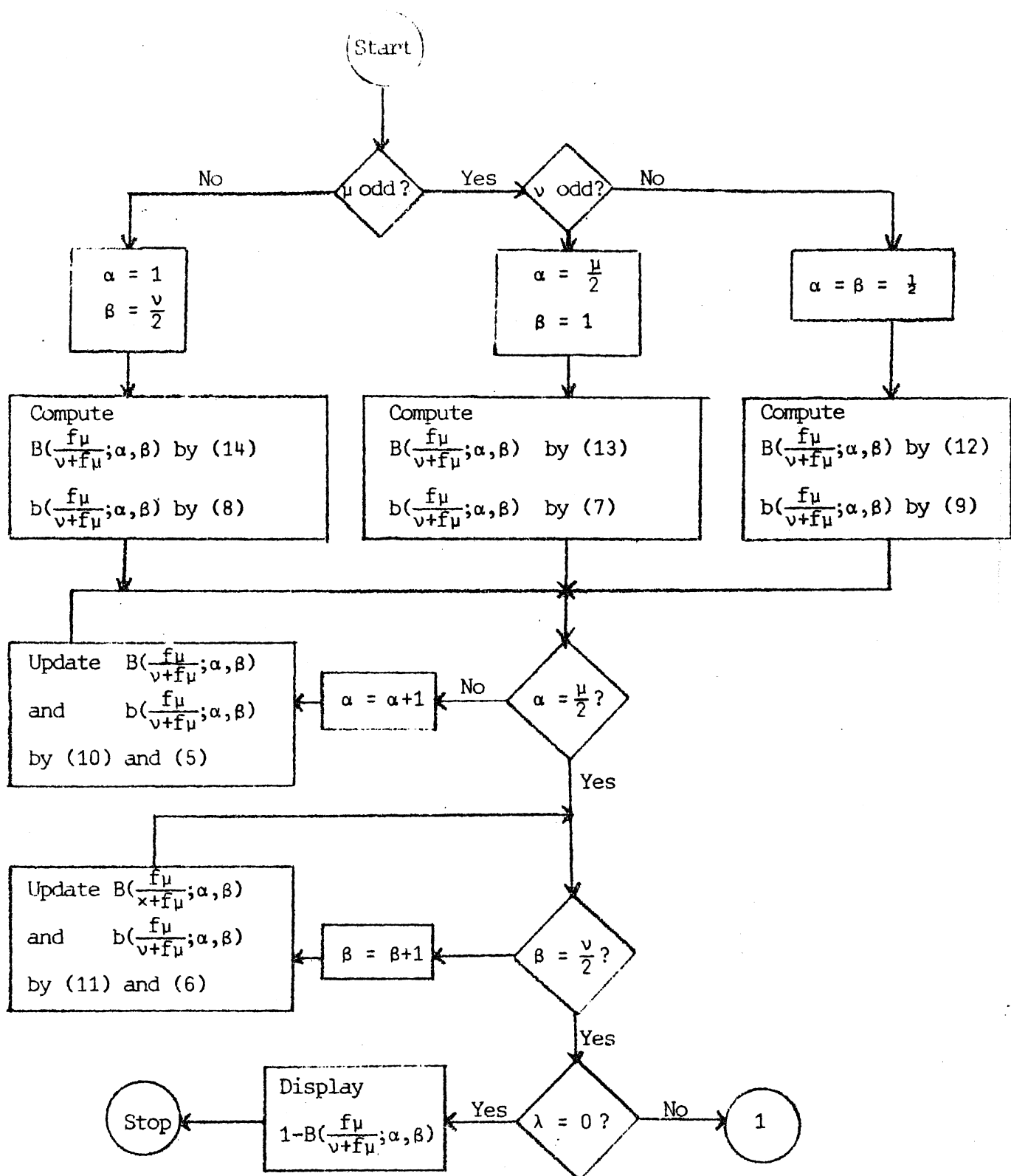
$$T_v(t) = \begin{cases} \frac{1}{2} [1 - G_{1, v}(t^2; 0)] & t \leq 0 \\ 1 - \frac{1}{2} [1 - G_{1, v}(t^2; 0)] & t \geq 0. \end{cases} \quad (20)$$

2. Algorithms for computing $1-G_{\mu,v}(f;\lambda)$ and $T_v(t)$

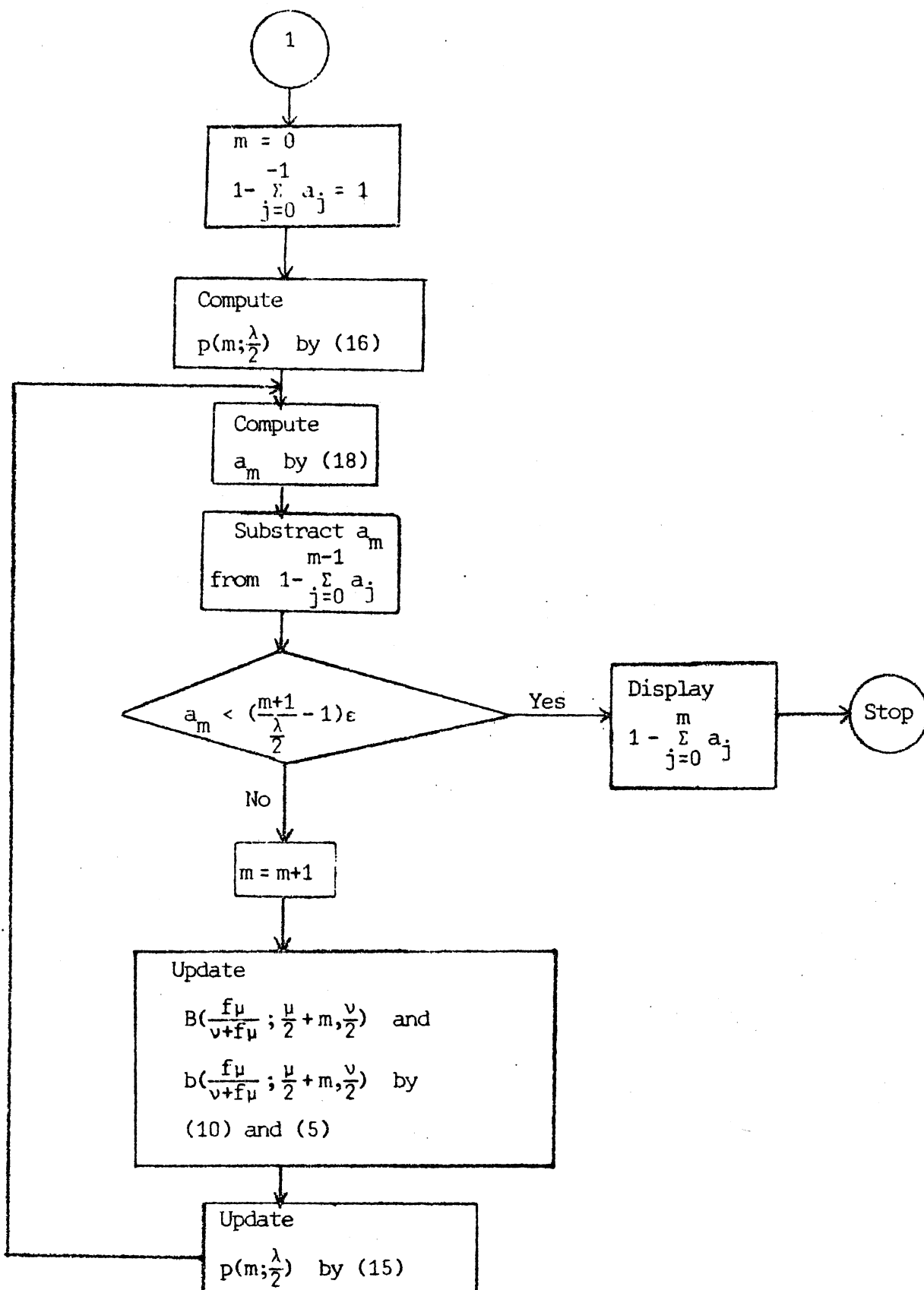
2A. The program in subsection 3 computes $1-G_{\mu,v}(f;\lambda)$ by the procedure shown in Flowcharts 1 and 2. Flowchart 1 shows how to compute $B(\frac{f\mu}{v+f\mu}; \frac{\mu}{2}, \frac{v}{2}) = G_{\mu,v}(f;0)$. If $\lambda > 0$, we proceed to Flowchart 2 showing how to compute $1 - \sum_{j=0}^m a_j$, where m is the smallest integer satisfying (19). Then according to Corollary 1 the approximation $1-G_{\mu,v}(f;\lambda) \approx 1 - \sum_{j=0}^m a_j$ gives an inaccuracy of less than ϵ .

If we are going to compute $1-G_{\mu,v}(f;\lambda)$ with a new value of λ , but the same μ, v , and f , we after having stored λ go directly to Flowchart 2 using the $B(\frac{f\mu}{v+f\mu}; \frac{\mu}{2}, \frac{v}{2})$ found in the previous computation.

2B. The program computes $T_v(t)$ by computing $1-G_{1,v}(t^2;0)$ by the procedure given in Flowchart 1 and using (20).



Flowchart 1.



Flowchart 2.

3. Program for $1 - G_{\mu, \nu}(f; \lambda)$ and $T_{\nu}(t; \lambda)$ on HP-67 calculator

3A. The program:

1 f LBL A	28 RCL B	55 RCL 6	82 STO÷1
2 enter	29 STO 7	56 RCL A	83 ÷
3 2	30 +	57 STO 9	84 -
4 ÷	31 STO÷6	58 STO 1	85 STO 0
5 STO B	32 STO÷7	59 h y ^x	86 f LBL 6
6 h x ^z y	33 RCL A	60 STO 0	87 RCL A
7 2	34 g FRAC	61 RCL 7	88 RCL 9
8 ÷	35 f x>0	62 ×	89 -
9 STO A	36 GTO 0	63 STO×1	90 f x=0
10 0	37 1	64 1	91 GTO 4
11 STO C	38 STO 9	65 STO 8	92 h ST I
12 h RTN	39 STO 0	66 GTO 6	93 f LBL 2
13 f LBL B	40 RCL 7	67 f LBL 1	94 RCL 9
14 enter	41 RCL B	68 RCL 6	95 STO÷1
15 2	42 STO 8	69 RCL 7	96 RCL 1
16 ÷	43 STO 1	70 ×	97 STO-0
17 STO C	44 h y ^x	71 f √	98 RCL 6
18 h CF 3	45 STO-0	72 STO 1	99 RCL 8
19 h RTN	46 RCL 6	73 .	100 RCL 9
20 f LBL C	47 ×	74 5	101 +
21 h F? 3	48 STO×1	75 STO 8	102 ×
22 GTO 9	49 GTO 6	76 STO 9	103 STO×1
23 GTO 5	50 f LBL 0	77 RCL 7	104 1
24 f LBL 9	51 RCL B	78 RCL 6	105 STO+9
25 RCL A	52 g FRAC	79 -	106 f DSZ
26 ×	53 f x>0	80 g sin ⁻¹	107 GTO 2
27 STO 6	54 GTO 1	81 h π	108 f LBL 4

109 RCL B	134 1	159 ÷	184 f LBL D
110 RCL 8	135 RCL 0	160 STO÷3	185 enter
111 -	136 -	161 1	186 2
112 f x=0	137 h RTN	162 STO+5	187 h 1/x
113 GTO 5	138 g LBL f a	163 STO+9	188 STO A
114 h ST I	139 CHS	164 -	189 ×
115 f LBL 3	140 g e ^x	165 EE X	190 STO B
116 RCL 8	141 STO 3	166 6	191 0
117 STO÷1	142 RCL 0	167 ÷	192 STO C
118 RCL 1	143 STO 8	168 g x>y	193 h RTN
119 STO + 0	144 RCL 1	169 GTO 8	194 f LBL E
120 RCL 7	145 STO 2	170 RCL 9	195 f x < 0
121 RCL 8	146 RCL A	171 STO÷2	196 h SF 2
122 RCL 9	147 1	172 RCL 2	197 g x ²
123 +	148 STO 4	173 STO-8	198 f GSB C
124 ×	149 STO 5	174 RCL 6	199 2
125 STO × 1	150 -	175 RCL B	200 ÷
126 1	151 STO 9	176 RCL 9	201 h F? 2
127 STO + 8	152 f LBL 7	177 +	202 h RTN
128 f DSZ	153 RCL 8	178 ×	203 CHS
129 GTO 3	154 RCL 3	179 STO × 2	204 1
130 f LBL 5	155 ×	180 GTO 7	205 +
131 RCL C	156 STO-4	181 f LBL 8	206 h RTN
132 f x > 0	157 RCL 5	182 RCL 4	
133 GTO f a	158 RCL C	183 h RTN	

Because of the use of formula (12) the calculator has to be in RAD mode.

3B. Computing $1 - G_{\mu, \nu}(f; \lambda)$

1. (Storing μ and ν .) Set: μ , enter, ν , A. In this operation $\lambda = 0$ is stored. Hence, if we are working with a central distribution, we may skip instruction 2 and proceed to instruction 3.

2. (Storing λ .) Set: λ , B.

3. (Computing $1 - G_{\mu, \nu}(f; \lambda)$.) Set: f , C. $1 - G_{\mu, \nu}(f; \lambda)$ is displayed.

New f , same μ, ν , and λ : Perform instruction 3.

New λ and f , same μ and ν : Perform instructions 2 and 3.

If we after having performed instruction 2 press B without entering f , the program leaves f unchanged from the previous computation.

Example 1. Set:

4, enter, 5, A;

3, B;

1.8927, C.

The program finds

$$1 - G_{4, 5}(1.8927; 3) = 0.46653.$$

Set:

4, B;

C.

The program finds

$$1 - G_{4, 5}(1.8927; 4) = 0.52812.$$

Set:

0, B;

C.

The program finds

$$1-G_{4,5}(1.8927;0)=0.25000.$$

Example 2. Set:

5,enter,4,A;

4.0506,C.

The program finds

$$1-G_{5,4}(4.0506;0)=0.10000.$$

Set:

3,B;

6.2560,C.

The program finds

$$1-G_{5,4}(6.2560;3)=0.10442.$$

Example 3. Set:

5,enter,3,A;

9.0135,C.

The program finds

$$1-G_{5,3}(9.0135;0)=0.05000.$$

Set:

2.4095,C.

The program finds

$$1-G_{5,3}(2.4095;0)=0.25000.$$

Set:

3,B;

C.

The program finds

$$1-G_{5,3}(2.4095;3)=0.39541.$$

The program uses $\epsilon = 10^{-6}$; this can be altered to $\epsilon = 10^{-n}$ by replacing 6 in program line 166 by n.

3C. Computing $T_v(t)$.

1. (Storing v.) Set: v,D.

2. (Computing $T_v(t)$.) Set: t,E. $T_v(t)$ is displayed.

New t, same v. Perform instruction 2.

Example 1. Set:

4,D;

0.7407,E.

The program finds

$T_4(0.7407)=0.75000$.

Set:

-0.7407,E.

The program finds

$T_4(-0.7407)=0.25000$.

Example 2. Set:

3,D;

1.6377,E.

The program finds

$T_3(1.6377)=0.90000$.

Set:

3.1824,E.

The program finds

$T_3(3.1824)=0.97500$.

3D. The registers have been used in the following way:

A $\mu/2$

B $\nu/2$

C $\lambda/2$

I DSZ

0 $B(\frac{f\mu}{\nu+f\mu}; \alpha, \beta); B(\frac{f\mu}{\nu+f\mu}; \frac{\mu}{2}, \frac{\nu}{2})$

1 $b(\frac{f\mu}{\nu+f\mu}; \alpha, \beta); b(\frac{f\mu}{\nu+f\mu}; \frac{\mu}{2}, \frac{\nu}{2})$

2 $b(\frac{f\mu}{\nu+f\mu}; \frac{\mu}{2} + m, \frac{\nu}{2})$

3 $p(m; \frac{\lambda}{2})$

4 $1 - \sum_{j=0}^m a_j$

5 m

6 $\frac{f\mu}{\nu+f\mu}$

7 $\frac{\nu}{\nu+f\mu}$

8 $\beta; B(\frac{f\mu}{\nu+f\mu}; \frac{\mu}{2} + m, \frac{\nu}{2})$

9 $\alpha; \frac{\mu}{2} + m$

Flags 2 and 3 and the labels A,B,C,D,E,a,0,1,2,3,4,5,6,7,8, and 9 have been used.

4. Related work

4A. An earlier version of the present paper appeared as Magnussen & Sundt (1977). That version also contained programs for $G_{\mu,\nu}(f;\lambda)$ on Hewlett-Packard HP-25 pocket calculator.. Those programs are also applicable on Hewlett-Packard HP-33 E pocket calculator.

Sverdrup (1976) describes how a program for the cumulative

eccentric Fisher distribution can also be used for calculations on the eccentric chi-square and Student distribution. He also gives programs on HP-25.

Guenther (1975) gives a computational routine for the cumulative eccentric Fisher distribution when the cumulative central Fisher distribution is pre-programmed.

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